Decision Diagrams for Linear Arithmetic

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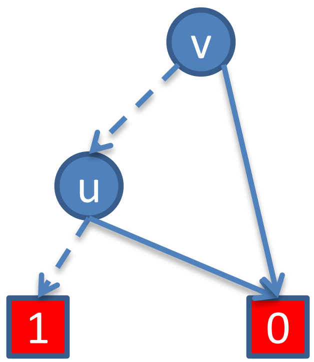
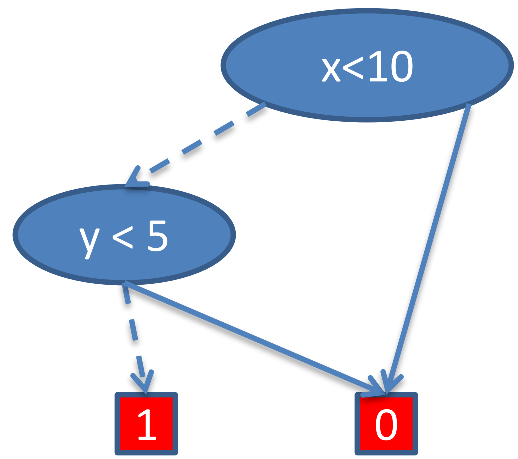
Linear Decision Diagrams (LDDs)

… are Binary Decision Diagrams with nodes labeled by linear inequalities

BDD



LDD



Our contributions:

- implementation on top of CUDD, including
    - support for propositional operations (AND, OR, NOT, ITE)
    - support for projection (i.e., existential quantification, QELIM) of numeric variables
    - dynamic variable ordering (DVO)
- benchmark and experiments

Motivation(1): Predicate + Numeric Abstractions

Predicate and Numeric Abstractions

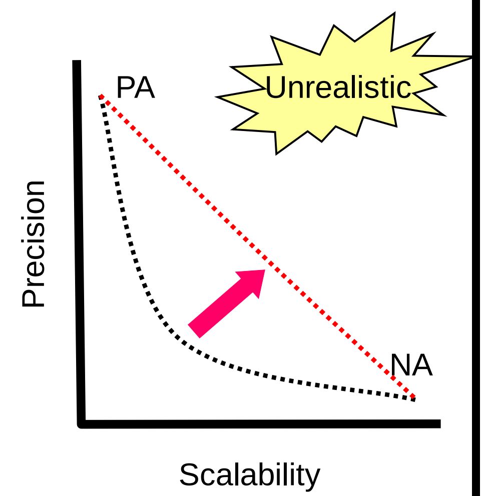
Predicate Abstraction (PA) (e.g., SDV)

- Typical property: no lock is acquired twice
- Program verification reduced to propositional reasoning with model checker
- Works well for control-driven programs
- Works poorly for data-driven programs

Numeric Abstraction (NA) (e.g, ASTREE)

- Typical property: no arithmetic overflow
- Program verification reduced to arithmetic reasoning
- Works well for data-driven programs
- Works poorly for control-driven programs

How to combine PA and NA to get the best of both?



Gurfinkel & Chaki. Combining Predicate and Numeric Abstraction for Software Model Checking. In FMCAD 2008

Motivation (2): Numeric Decision Diagrams

Numeric Decision Diagrams

NDD elements are

$2^P \rightarrow 2^N$

BDDs over Predicate and Numeric Terms

$(p_1 \&\& p_2) \ ||$

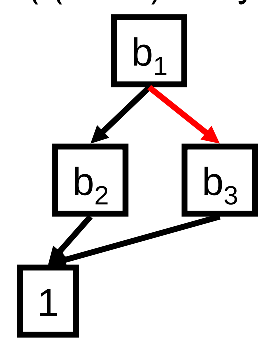
$(x<0 \ \&\& \ y=z)$

normalize

$(x>=0 \ \&\& \ z>0) \ ||$

$(!(x>=0) \ \&\& \ y=z)$

p_1: x>=0, p_2: z>0



1-edges are black, 0-edges are red

edges to 0 node are not shown

b1:x>=0, b2:z>0, b3:y=z

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Combining PA and NA for Soft MC
Gurfinkel and Chaki

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Motivation (2): Numeric Decision Diagrams

Numeric Decision Diagrams

Problems with NDDs are:

- No reduction w.r.t. the types of constraints
- All numeric operations are done path-at-a-time (i.e., exponential in the diagram!!!)

Lesson learned: need diagrams for linear arithmetic with efficient (not path-at-a-time) existential quantification

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Some Other Applications of LDDs

- Represent and manipulate Boolean formulas over linear arithmetic …
  - to compute predicate abstraction
  - to summarize loop-free code
  - for program analysis with disjunctive abstract domain
  - to combine predicate and numeric abstractions
  - for timed automata verification
  - …
- LDDs are NOT good for SATISFIABILITY checking
  - not a substitute for an SMT solver

Talk Outline

- The basics
  - variable ordering, reduction rules, propositional operations
- Dynamic Variable Ordering
- Quantifier Elimination
  - existential quantification of a single variable
  - heuristics for quantifying multiple variables
- Implementation, Benchmarks, Results
- Conclusion and Future Work

Canonizing Linear Inequalities

coefficient variable constant

Linear Inequality: $\underbrace{a_1\, x_1 + \ldots + a_n\, x_n}_{\text{term}} \leq k$

$x < 10$	$\equiv$	$x \leq 9$
$x + y = 10$	$\equiv$	$x + y \leq 10 \wedge x + y \geq 10$
$x + y \geq 10$	$\equiv$	$-x - y \leq -10$
$-x-y \leq -10$	$\equiv$	$\neg\,(x + y \leq 9)$

LDD Node Ordering

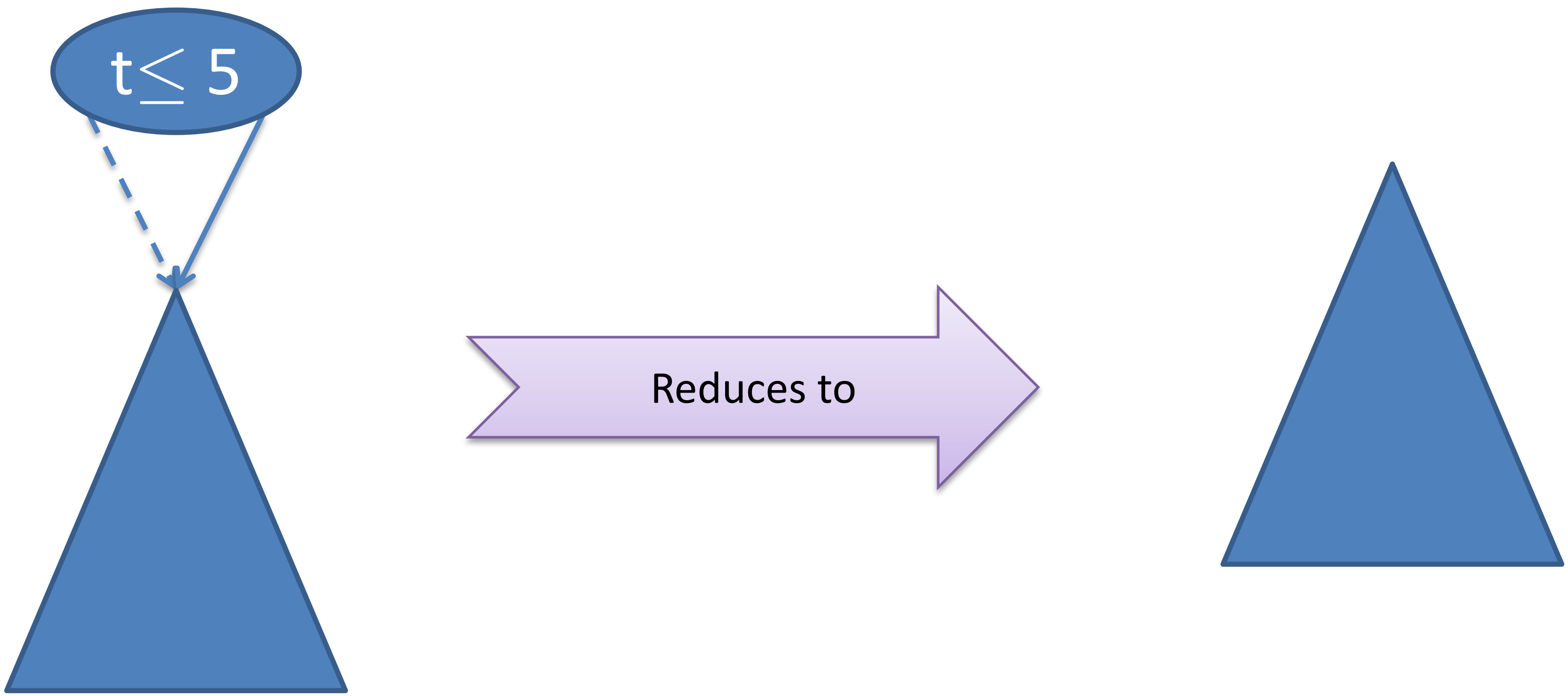
Random: $\{x \leq 0\}$ $\{x \leq 10\}$ $\{y \leq 5\}$ $\{x \leq 5\}$ $\{z \leq 6\}$ $\{y \leq 3\}$

Term-sorted: $\{x \leq 0\}$ $\{x \leq 10\}$ $\{x \leq 5\}$ $\{y \leq 5\}$ $\{y \leq 3\}$ $\{z \leq 6\}$

Ordered: $\{x \leq 0\}$ $\{x \leq 5\}$ $\{x \leq 10\}$ $\{y \leq 3\}$ $\{y \leq 5\}$ $\{z \leq 6\}$

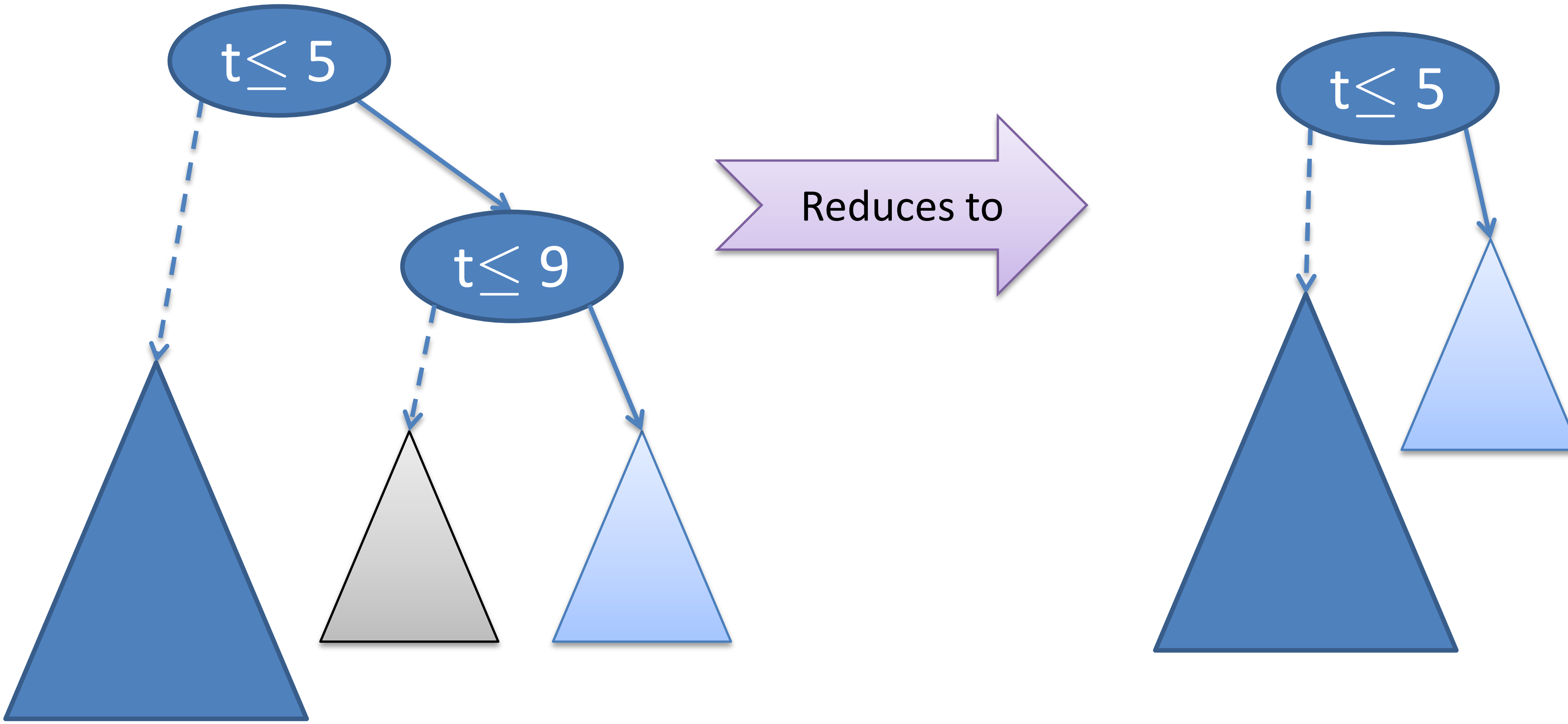
Ordered: $\{y \leq 3\}$ $\{y \leq 5\}$ $\{z \leq 6\}$ $\{x \leq 0\}$ $\{x \leq 5\}$ $\{x \leq 10\}$

Reduction: Different Children

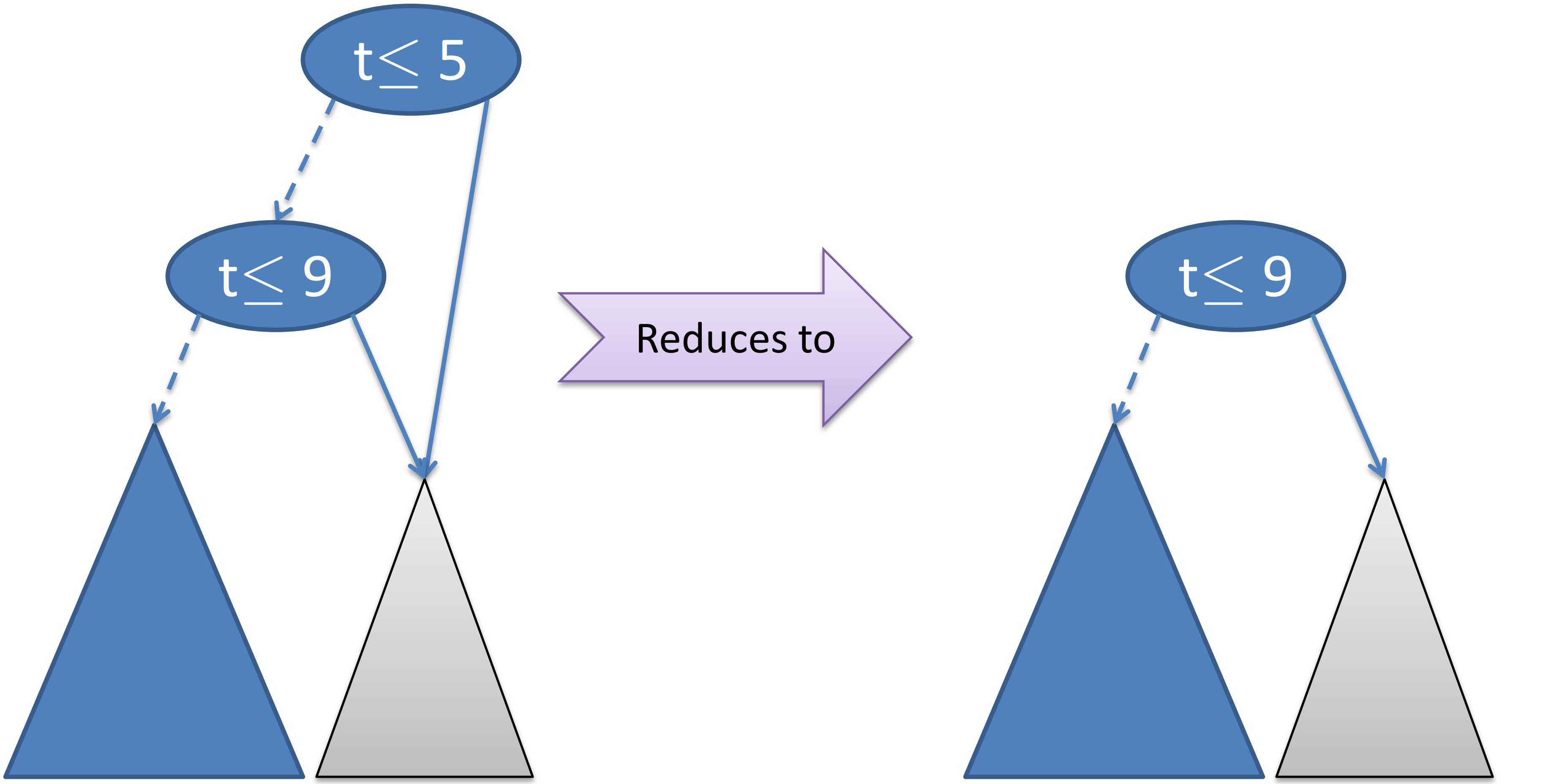


Same as BDD

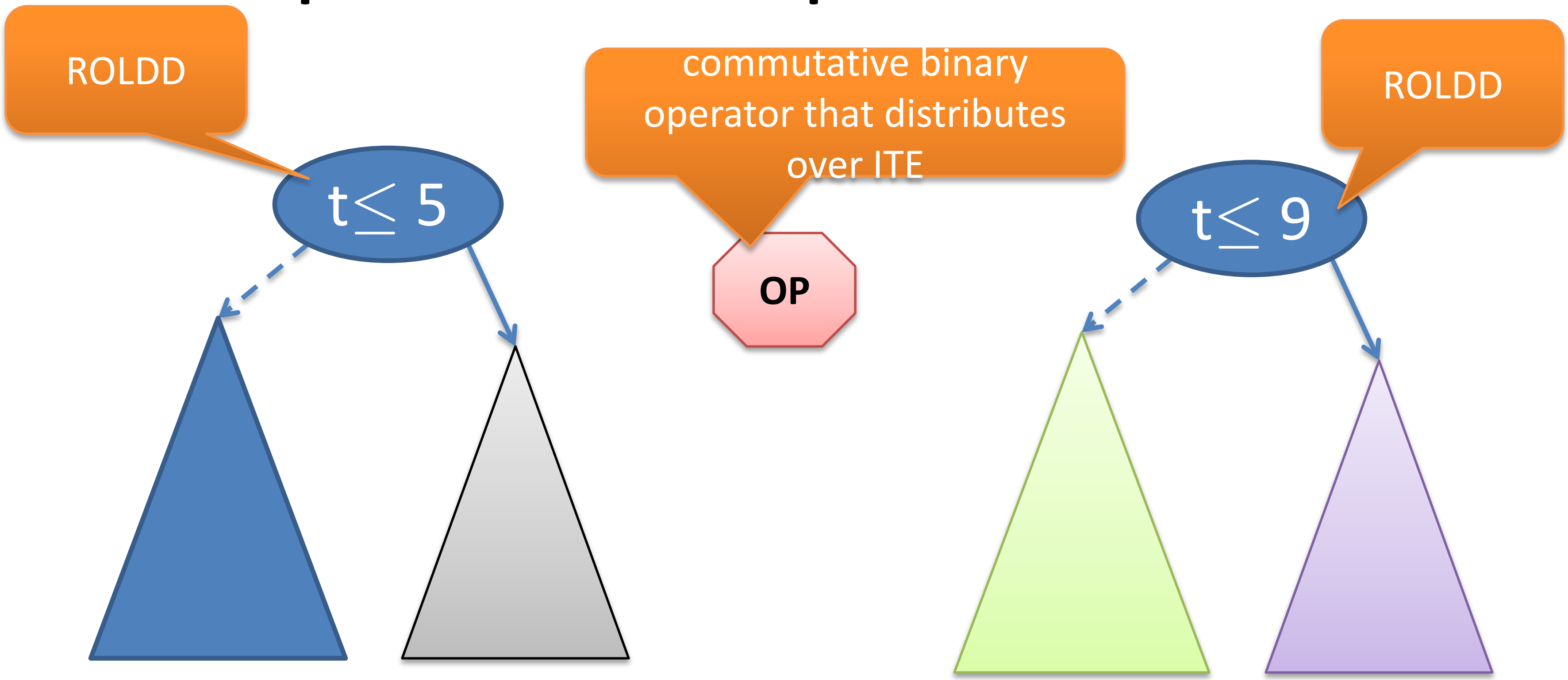
Reduction: Imply High



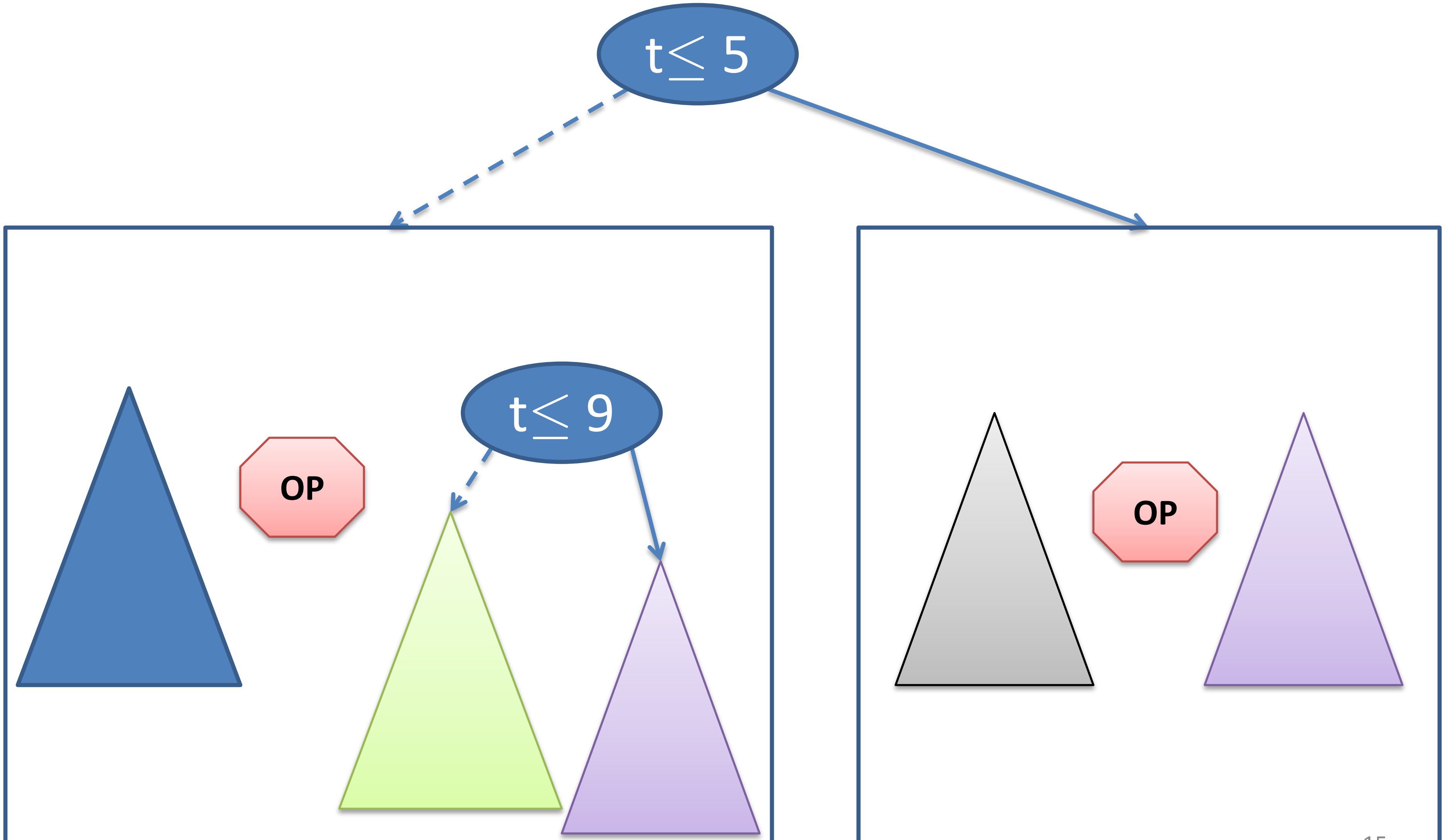
Reduction: Imply Low



Propositional Operations: APPLY

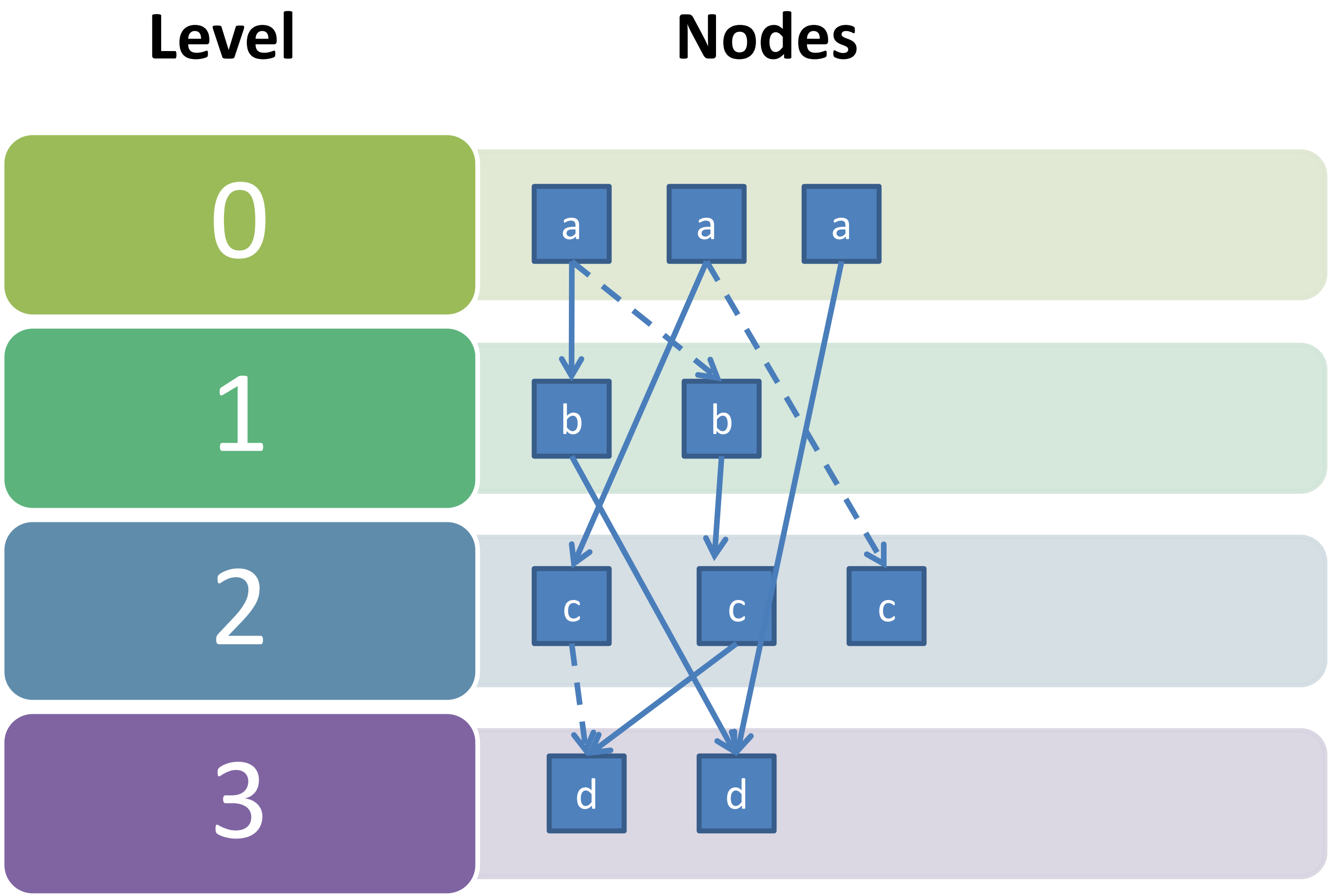


Propositional Operations: APPLY



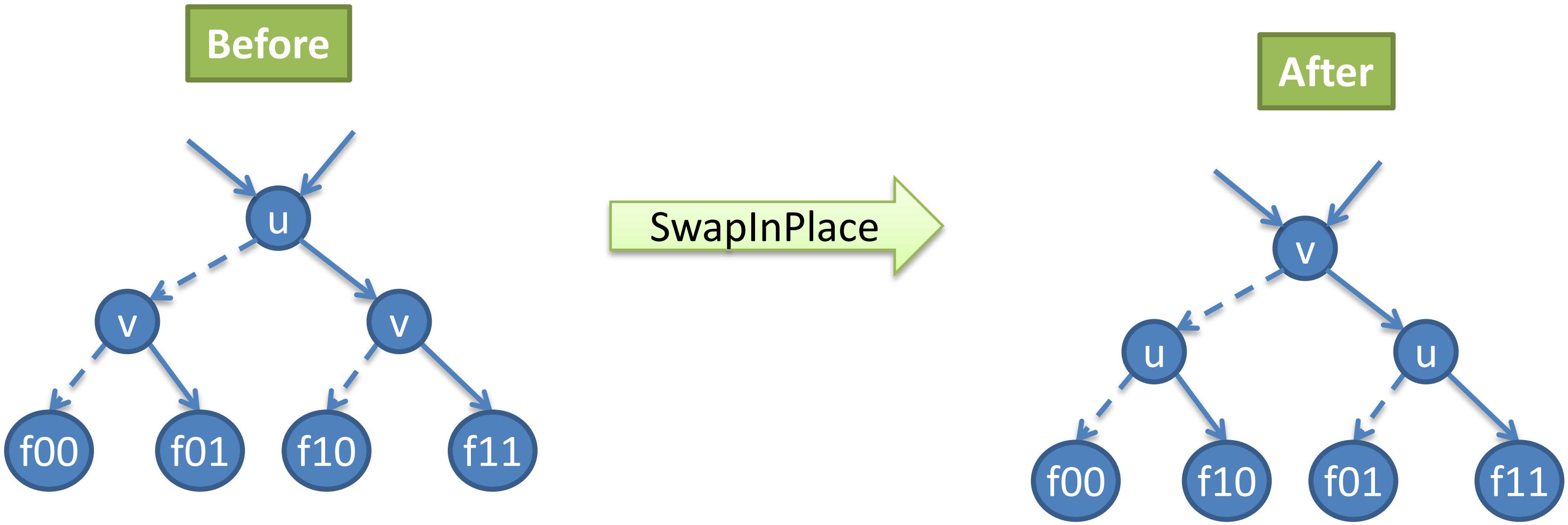
Rudell’s DVO Algorithm for BDDs

Unique Table



*Edges to 0 and 1 are not shown

Example: Changing levels



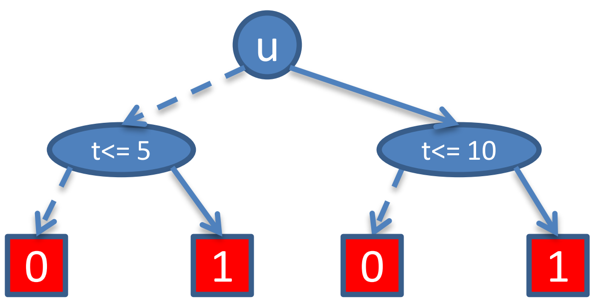
(u, (v, f11, f10), (v, f01, f00)) is overwritten in place by (v, (u, f11, f01), (u, f10, f00))
Trivial new cofactors are reduced, i.e., when f00=f10 or f01=f11
Only the diagram rooted at *u* is changed (both the label and the children are new)

Complexity: linear in the number of nodes labeled with u in the unique table

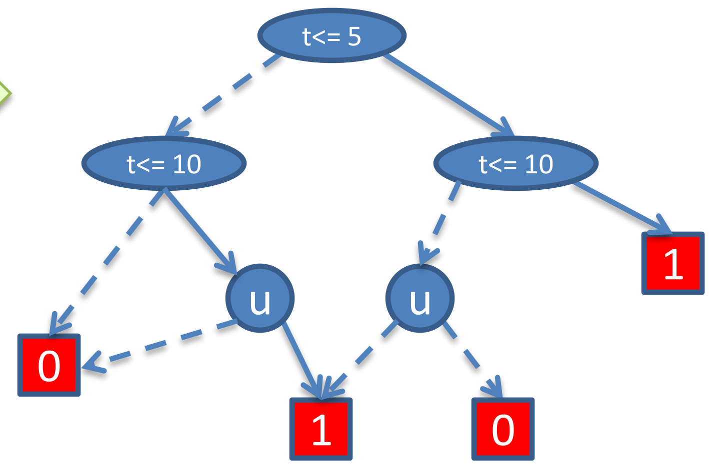
Example: Changing levels in ROLDD

Before Reordering

After Reordering



SwapInPlace



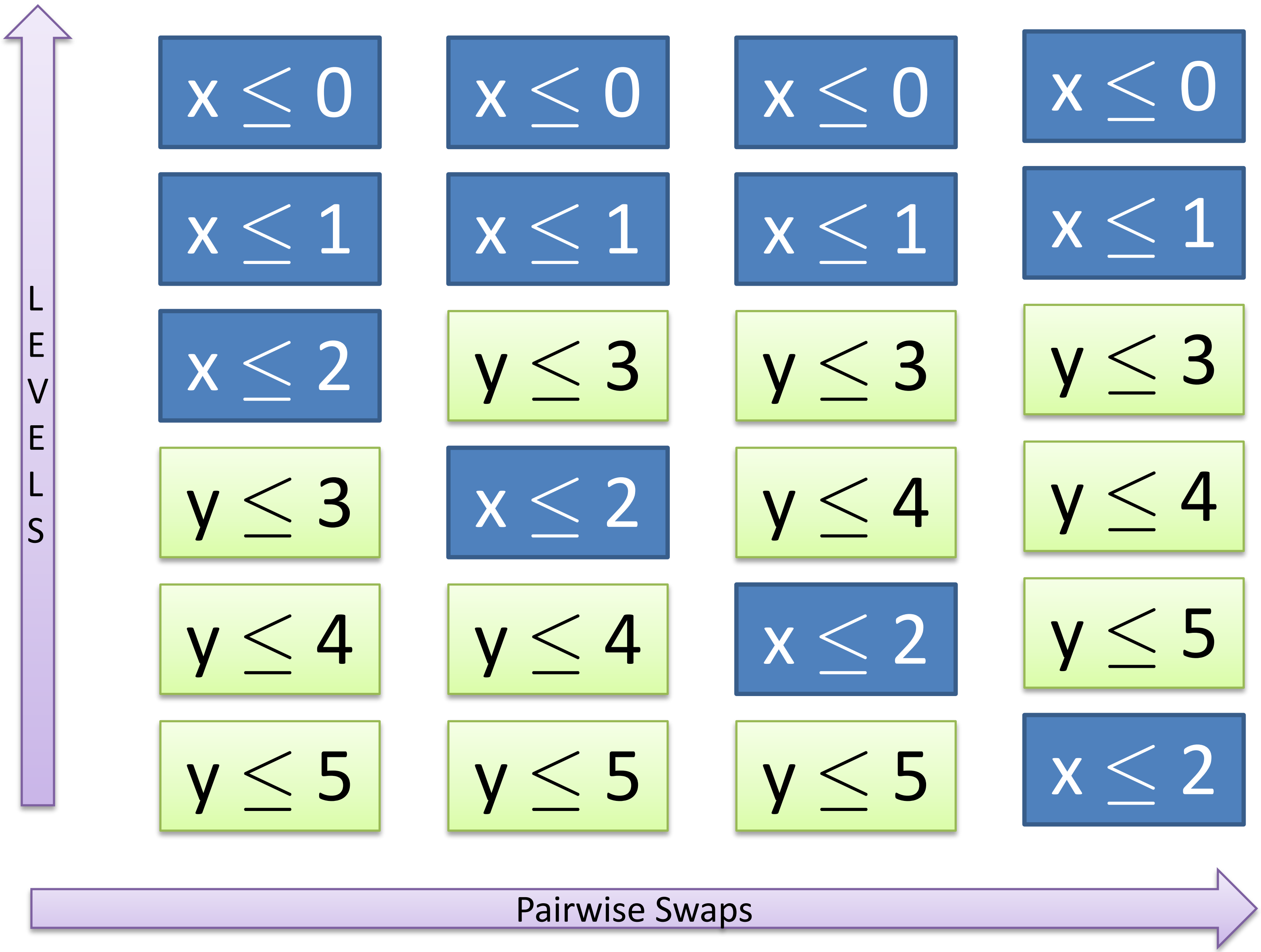
ROLDD with order: u, t<=5, t<=10
(shown as a tree)

New order: t<=5, t<=10,u
Not reduced!

Cannot use BDD reordering for LDD!

Problems extending DVO to LDDs

- Broken ordering constraints
  - **Solution: swap adjacent terms instead of adjacent levels**
- Broken Imply-high and Imply-low rules
  - **Solution: enforce the rules in swapInPlace**
- LDDs are not canonical
  - **Solution: LDDs are "canonical enough": in SwapInPlace the root node is never reduced**


LEVELS
$x \leq 0$
$x \leq 0$
$x \leq 0$
$x \leq 0$
$x \leq 1$
$x \leq 1$
$x \leq 1$
$x \leq 1$
$x \leq 2$
$y \leq 3$
$y \leq 3$
$y \leq 3$
$y \leq 3$
$x \leq 2$
$y \leq 4$
$y \leq 4$
$y \leq 4$
$y \leq 4$
$x \leq 2$
$y \leq 5$
$y \leq 5$
$y \leq 5$
$y \leq 5$
$x \leq 2$
Pairwise Swaps

Two techniques for QELIM

- **Black Box:** use QELIM for conjunctions as a black box. Apply it to all paths of a diagram
  - linear in the number of paths == exponential in the size of the diagram!
  - many examples in the literature. (e.g., we used it in NDDs)
- **White Box:** Extend Fourier-Motzkin QELIM to the DAG of LDD
  - in the best case, same complexity as BDD quantification

Fourier-Motzkin QELIM

FM1(Var *y*, Conjunction φ)
  let *S* be all constraints with *y*
  remove *S* from φ
  add all pairwise resolutions of *S* to φ

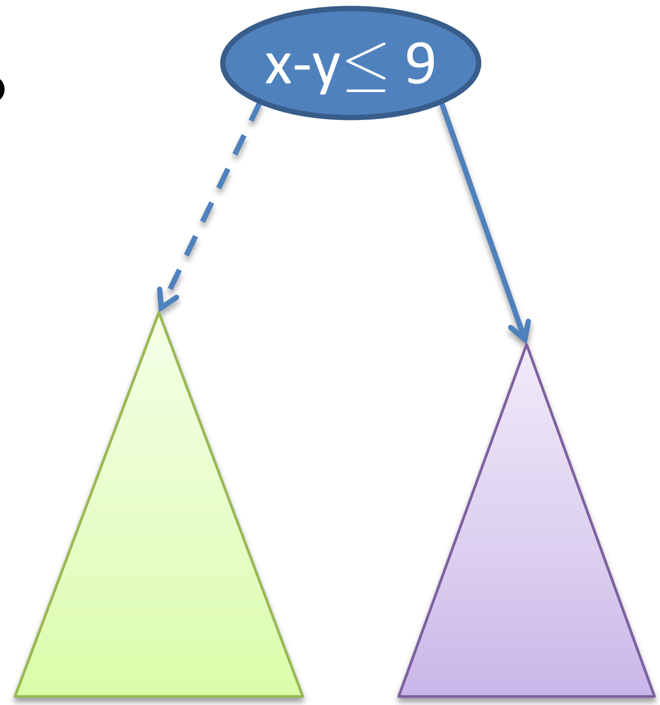
$$\exists\, y\, .\, \boxed{x\text{-}y \leq 5} \;\wedge\; x\text{-}z \geq 8 \;\wedge\; \boxed{y\text{-}z \leq 10}$$

$$x\text{-}z \geq 8 \wedge x\text{-}z \leq 15$$

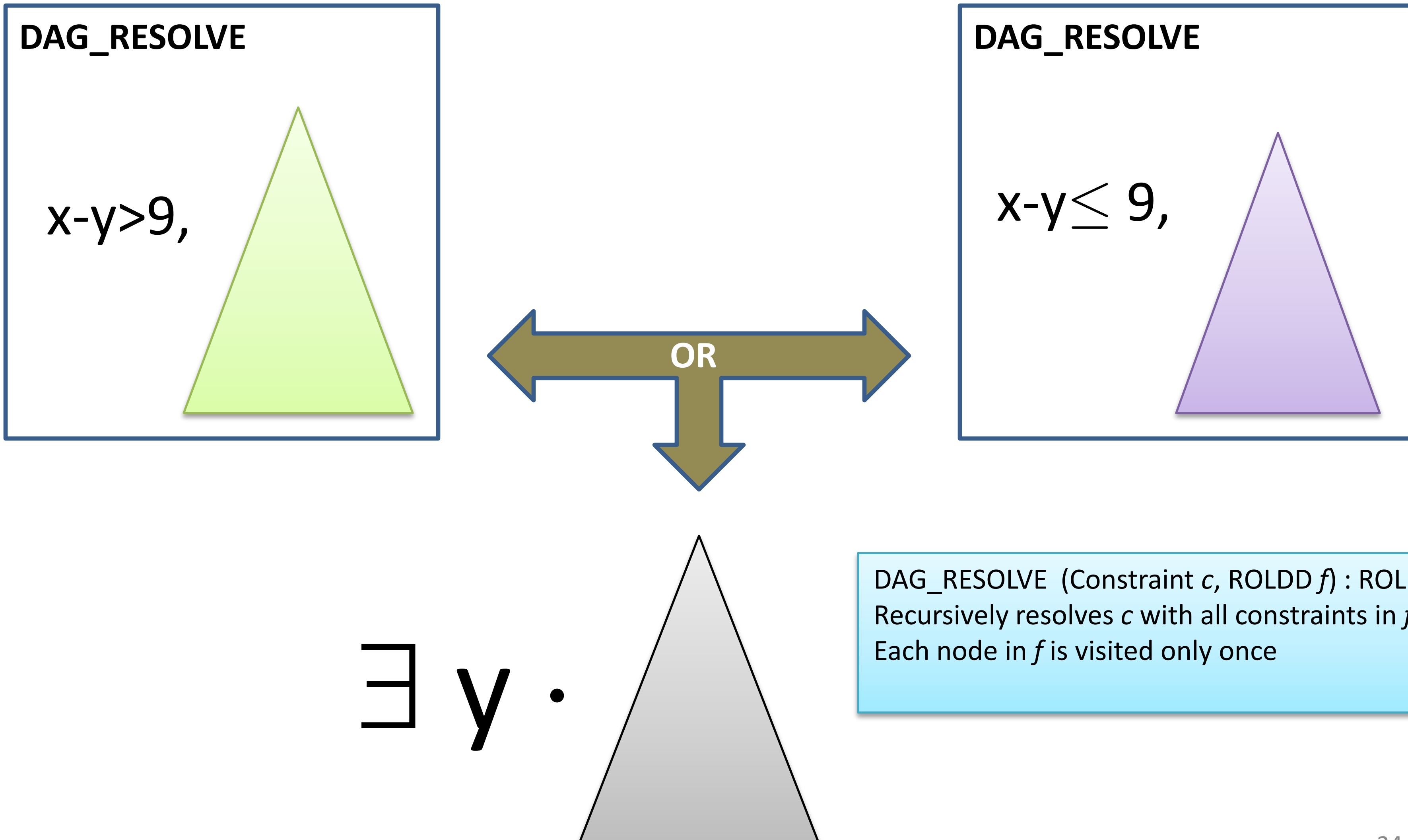
FM2(Var *y*, Formula φ)
  while exists constraint *c* with *y* in φ **do**
    remove *c* from φ
    resolve *c* with remaining constraints in φ
  end while

WB_EXISTS1: Example

$\exists\, y \cdot$



WB_EXISTS1: Example (Cont)



DAG_RESOLVE (Constraint *c*, ROLDD *f*) : ROLDD
Recursively resolves *c* with all constraints in *f*
Each node in *f* is visited only once

Quantifying out multiple variables

```
EXISTS(LDD f, Vars V)
  res = f;
  while (V != empty)
    V' = FIND_DROP_VARS (V, res);
    if (V' != empty)
      res = DC(CONS_OF(V'), res);
      V = V \ V';
      continue;

      u = PICK_VAR (V, res);
      res = WB_EXISTS1(u, res);
      V = V \ {u};
  end while
  return res;
```

EXISTS1 -- any quantification procedure that can eliminate a single variable. In our implementation, it is the optimized WB_EXISTS1 from previous slides

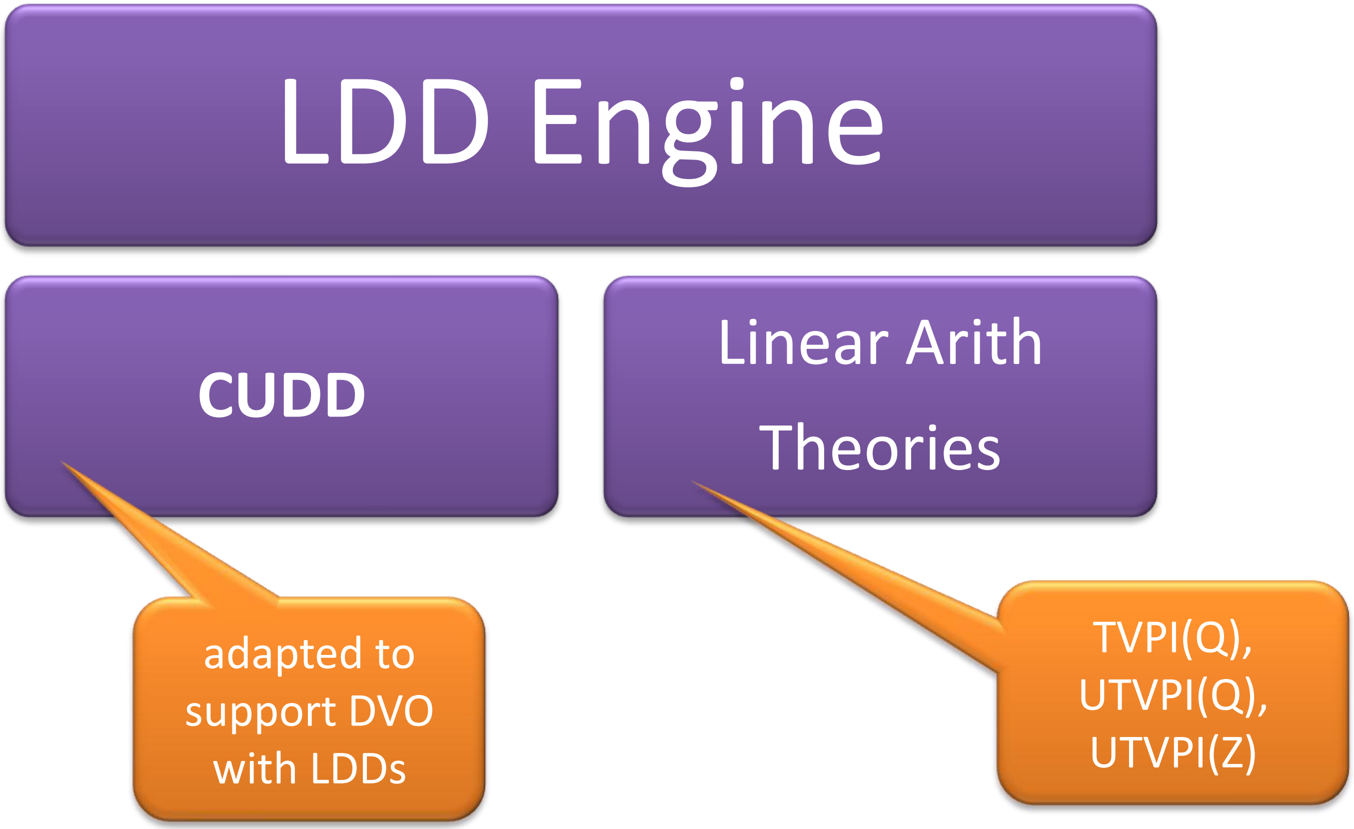
DC short for DROP_CONS

FIND_DROP_VARS(V, res) – finds all variables in V that have trivial resolutions on all 1-paths of res

PICK_VAR (V, res) -- picks a variable from V to be quantified out next

In our implementation, FIND_DROP_VARS and PICK_VAR are based on looking at the set of all constraints that are in support of res.

The Implementation



Benchmark: Image Computation

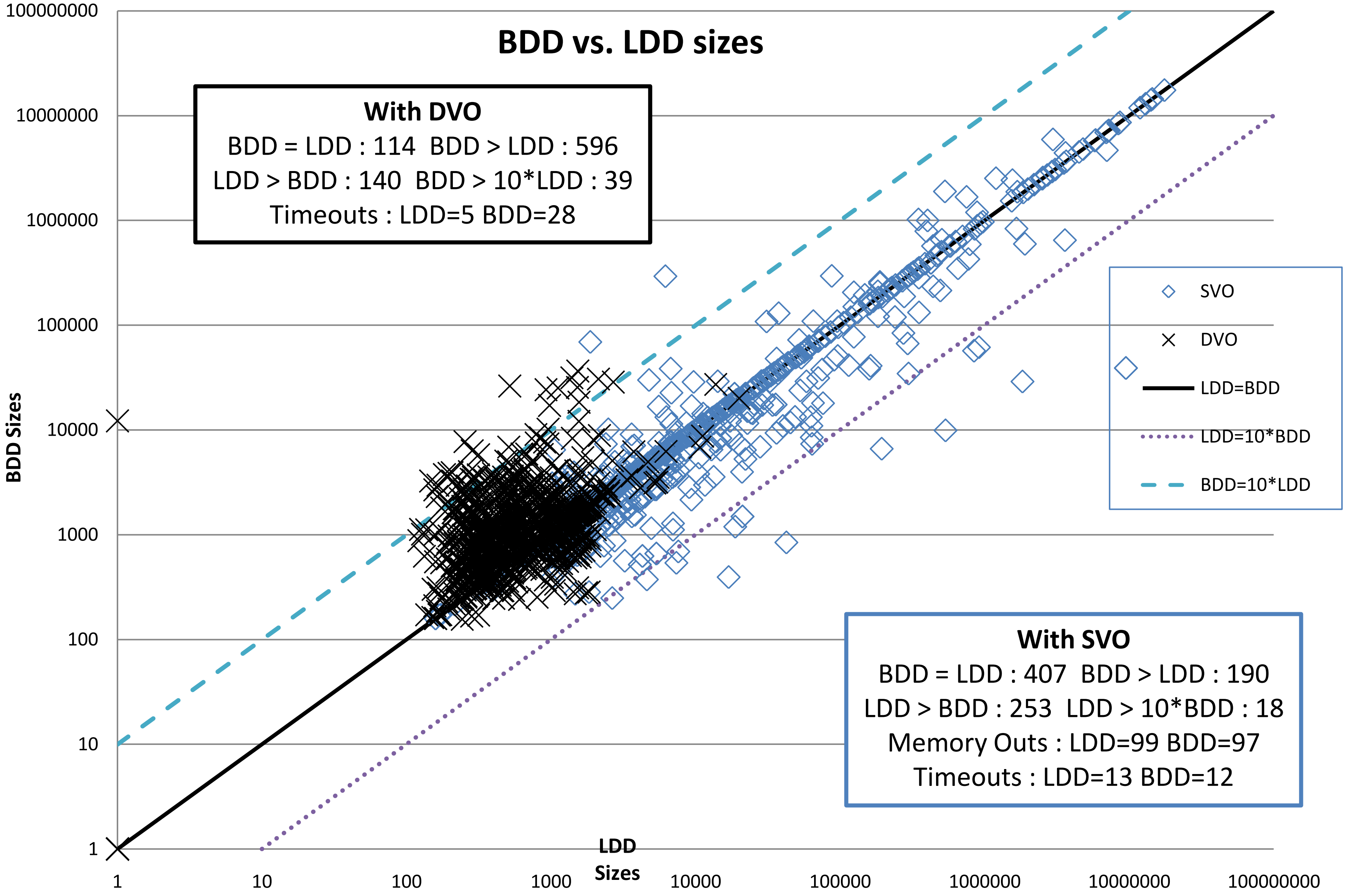
Test case: $\exists V \cdot R(V, V')$

transition relation of a loop-free program fragment

- Each test case is constructed
  - from open source software: CUDD, mplayer, bzip2,...
  - extracted using LLVM into SSA with optimizations, aggressive loop-unrolling, and inlining
  - approximated using UTVPI constraints
- Stats: 850 test cases

  4KB – 700KB (in SMT-LIB format),

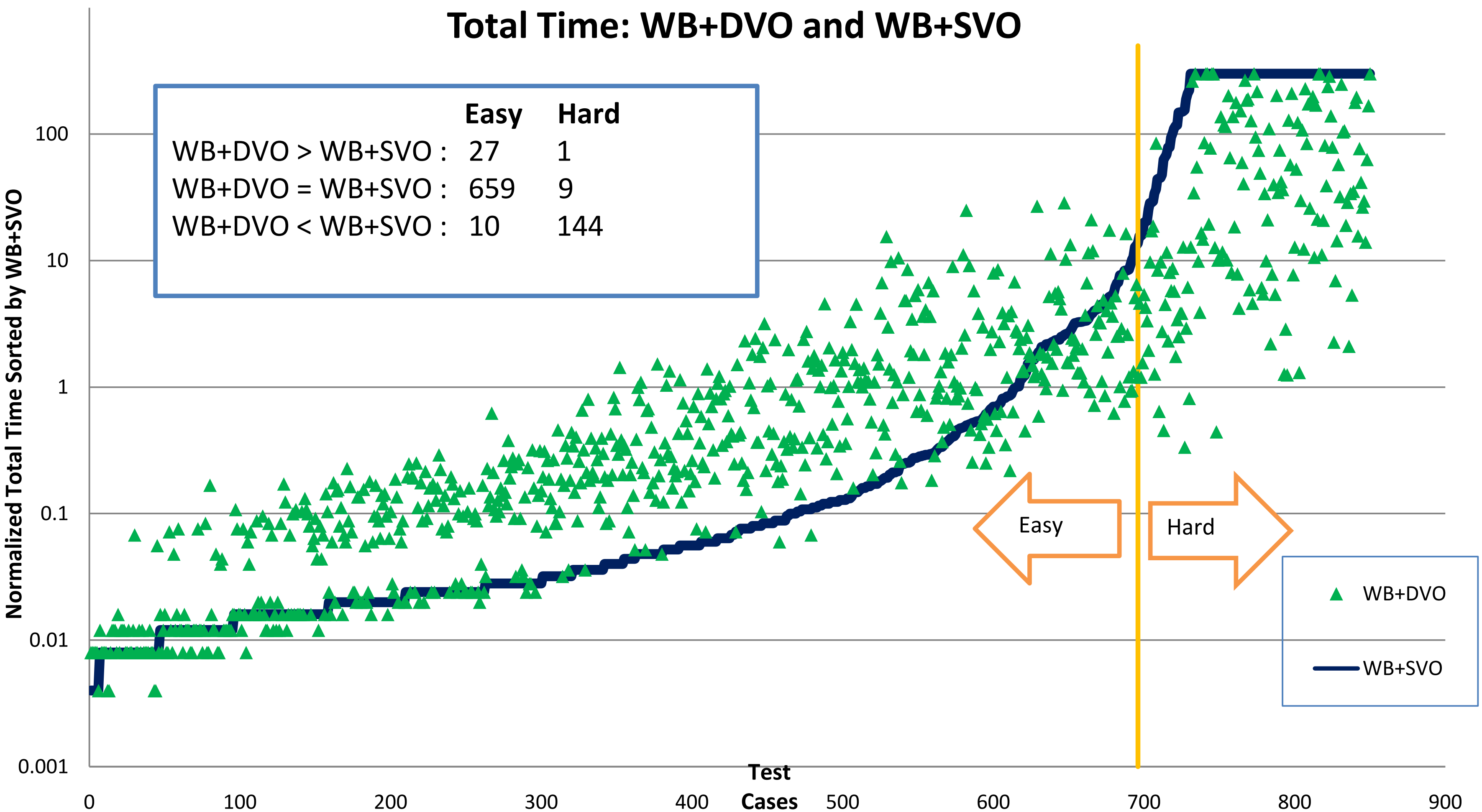
  30 – 7,956 variables



BDD vs. LDD sizes
With DVO
BDD = LDD : 114 BDD > LDD : 596
LDD > BDD : 140 BDD > 10*LDD : 39
Timeouts : LDD=5 BDD=28
With SVO
BDD = LDD : 407 BDD > LDD : 190
LDD > BDD : 253 LDD > 10*BDD : 18
Memory Outs : LDD=99 BDD=97
Timeouts : LDD=13 BDD=12
SVO
DVO
LDD=BDD
LDD=10*BDD
BDD=10*LDD
BDD Sizes
LDD Sizes
100000000
10000000
1000000
100000
10000
1000
100
10
1

Overall Results for QELIM

	Hard (154 cases)				Easy (696 cases)		
Alg.	***Total (sec)***	***QE (sec)***	***TO***	***MO***	***Total (sec)***	***QE (sec)***	***TO***
BB	--	--	141	0	--	--	670
WB+SVO	38,739	36,511	21	99	395	80	0
WB+DVO	10,953	3,329	9	0	784	219	0


Total Time: WB+DVO and WB+SVO
Easy Hard
WB+DVO > WB+SVO : 27 1
WB+DVO = WB+SVO : 659 9
WB+DVO < WB+SVO : 10 144
Normalized Total Time Sorted by WB+SVO
100
10
1
0.1
0.01
0.001
0
100
200
300
400
500
600
700
800
900
Test Cases
Easy
Hard
WB+DVO
WB+SVO

Predicate Abstraction with LDDs

$$\exists V \cdot R(V) \wedge \bigwedge_i (b_i \leftrightarrow p_i(V))$$

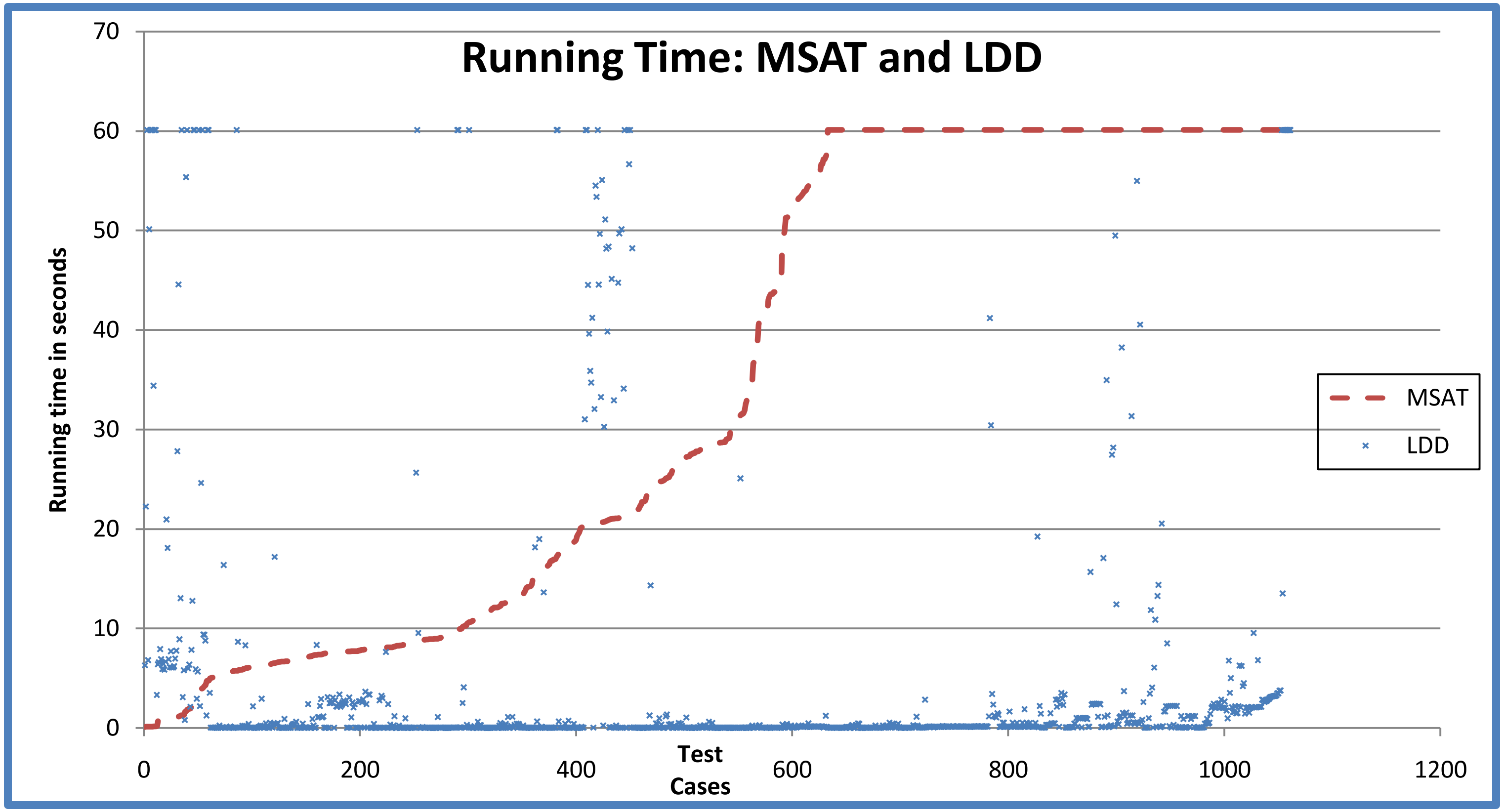
transition relation of a loop-free program fragment

Boolean variable

predicate

Predicate Abstraction with LDDs

$$\exists V \cdot R(V) \wedge \bigwedge_i (b_i \leftrightarrow p_i(V))$$



Related Work

Decision Diagrams (over linear constraints)

- Strehl. *Interval Diagram Techniques…* 1999
- Moller et al. *Difference Decision Diagrams.* 1999
- Larsen et al. *Clock Difference Diagrams.* 1999

Quantifier Elimination in Large Boolean Formulas

- Clarke et al. *SATABS: A SAT-Based PA for ANSI-C*. 2005
- Lahiri et al. *SMT Techniques for Fast PA.* 2006
- Cavada et al. *Computing PA by Integrating BDDs and SMT Solver. 2007*
- D. Monniaux. *A QELIM Algorithm for Linear Real Arith. 2008*

Future Work

- Predicate Abstractions with LDDs
- An LDD-based Abstract Domain
  - first step is a disjoint-box domain for variable range analysis
  - designing a widening is the main challenge
- Public release of the library
  - send email to arie@cmu.edu for more info

THE END